A CONVERGENCE ON BOOLEAN ALGEBRAS GENERALIZING THE CONVERGENCE ON THE ALEKSANDROV CUBE

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Abstract

We compare the forcing related properties of a complete Boolean algebra $\mathbb{B}$ with the properties of the convergences $\lambda_{\rm s}$ (the algebraic convergence) and $\lambda_{\rm ls}$ on $\mathbb{B}$ generalizing the convergence on the Cantor and Aleksandrov cube respectively. In particular we show that $\lambda_{\rm ls}$ is a topological convergence iff forcing by $\mathbb{B}$ does not produce new reals and that $\lambda_{\rm ls}$ is weakly topological if $\mathbb{B}$ satisfies condition $(\hbar)$ (implied by the $\mathfrak{t}$-cc). On the other hand, if $\lambda_{\rm ls}$ is a weakly topological convergence, then $\mathbb{B}$ is a $2^{\mathfrak{h}}$-cc algebra or in some generic extension the distributivity number of the ground model is greater than or equal to the tower number of the extension. So, the statement "The convergence $\lambda_{\rm ls}$ on the collapsing algebra $\mathbb{B} = \mathrm{ro}(^{<\omega}\omega_2)$ is weakly topological" is independent of ZFC.

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1 Introduction

The object of our study is the interplay between the forcing related properties of a complete Boolean algebra $\mathbb{B}$ and the properties of convergence structures defined on $\mathbb{B}$. In Section 3 we observe the algebraic convergence $\lambda_{\rm s}$, generalizing the convergence on the Cantor cube and generating the sequential topology $\mathcal{O}_{\lambda_{\rm s}}$ introduced by Maharam and investigated in the context of the von Neumann and Maharam's Problem. In the rest of the paper we investigate the convergence $\lambda_{\rm ls}$, introduced in Section 4 as a natural generalization of the convergence on the Aleksandrov cube.

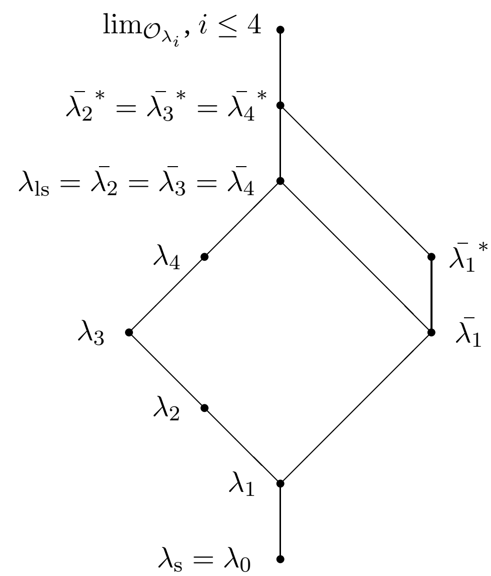
Concerning the context of our research, first we note that the topology $\mathcal{O}_{\lambda_{\rm ls}}$ (generated by the convergence $\lambda_{\rm ls}$) and its dual $\mathcal{O}_{\lambda_{\rm li}}$ generate the sequential topology $\mathcal{O}_{\lambda_{\rm s}}$, for the algebras $\mathbb{B}$ belonging to a wide class including Maharam algebras [12]. Second, we mention some related results. If $\mathbb{B}$ is a complete Boolean algebra, let the convergences $\lambda_i : \mathbb{B}^\omega \to P(\mathbb{B})$, for $i \in \{0,1,2,3,4\}$, be defined by

$$\lambda_i(x) = \begin{cases} \{b_4(x)\} & \text{if } b_i(x) = b_4(x), \\ \emptyset & \text{if } b_i(x) < b_4(x), \end{cases}$$

where $x = \langle x_n \rangle \in \mathbb{B}^\omega$, $\tau_x = \{\langle \check{n}, x_n \rangle : n \in \omega\}$ is the corresponding $\mathbb{B}$-name for a subset of ω and

$$\begin{aligned}
b_0(x) &= \|\tau_x \text{ is cofinite}\| = \liminf x,\\
b_1(x) &= \|\tau_x \text{ is old infinite}\|,\\
b_2(x) &= \|\tau_x \text{ contains an old infinite subset}\|,\\
b_3(x) &= \|\tau_x \text{ is infinite and non-splitting}\|,\\
b_4(x) &= \|\tau_x \text{ is infinite}\| = \limsup x.
\end{aligned}$$

Then, by [11] and [12], $\lambda_{\mathrm{s}} = \lambda_0$ and $\lambda_{\mathrm{ls}} = \bar{\lambda}_2 = \bar{\lambda}_3 = \bar{\lambda}_4$, where $\bar{\lambda}$ is the closure of a convergence λ under (L2). Also $\lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \lambda_4$ and these four convergences generate the same topology, $\mathcal{O}_{\lambda_{\mathrm{ls}}}$. So we have the following diagram (λ^* denotes the closure of a convergence λ under (L3), see Section 2).



Now we mention some related results from [11] and [12]. The property that $\mathbb{B}$ does not produce new reals by forcing is equivalent to each of the following conditions: $\lambda_1 = \lambda_2$, $\lambda_1 = \lambda_4$, $\lambda_2 = \lambda_3$, $\lambda_2 = \lambda_4$, $\bar{\lambda}_1 = \lambda_{\mathrm{ls}}$, $\bar{\lambda}_1$ is a topological convergence. The property that $\mathbb{B}$ does not produce splitting reals is equivalent to the equality $\lambda_3 = \lambda_4$, which holds if the convergence $\bar{\lambda}_1$ is weakly topological.

Our notation is mainly standard. So, ω denotes the set of natural numbers, Y^X the set of all functions $f : X \to Y$ and $\omega^{\uparrow\omega}$ the set of all strictly increasing functions from ω into ω. By $|X|$ we denote the cardinality of the set X and, if κ is a cardinal, then $[X]^\kappa = \{A \subset X : |A| = \kappa\}$ and $[X]^{<\kappa} = \{A \subset X : |A| < \kappa\}$. By

$\mathfrak{c}$ we denote the cardinality of the continuum. For subsets A and B of ω we write $A \subset^* B$ if $A \setminus B$ is a finite set and $A \subsetneq^* B$ denotes $A \subset^* B$ and $B \not\subset^* A$. The set S **splits** the set A if the sets $A \cap S$ and $A \setminus S$ are infinite. $\mathcal{S} \subset [\omega]^\omega$ is called a **splitting family** if each set $A \in [\omega]^\omega$ is split by some element of $\mathcal{S}$ and $\mathfrak{s}$ is the minimal size of a splitting family (the **splitting number**). A set P is a **pseudointersection** of a family $\mathcal{T} \subset [\omega]^\omega$ if $P \subset^* T$ for each $T \in \mathcal{T}$. A family $\mathcal{T} \subset [\omega]^\omega$ is a **tower** if $\langle \mathcal{T}, {}^*\!\supsetneq \rangle$ is a well-ordered set and $\mathcal{T}$ has no pseudointersection. The **tower number**, $\mathfrak{t}$, is the minimal size of a tower in $[\omega]^\omega$. If $\langle \mathbb{P}, \leq \rangle$ is a partial order, a subset $D \subset \mathbb{P}$ is called **dense** if $\forall p \in \mathbb{P}\ \exists d \in D\ d \leq p$ and D is called **open** if $p \leq q \in D$ implies $p \in D$. The **distributivity number**, $\mathfrak{h}$, is the minimal size of a family of dense open subsets of the order $\langle [\omega]^\omega, \subset \rangle$ whose intersection is not dense. More information on invariants of the continuum the reader can find in [5].

If $\mathbb{B}$ is a Boolean algebra and $A \subset \mathbb{B}$ let $A\uparrow = \{b \in \mathbb{B} : \exists a \in A\ a \leq b\}$; instead of $\{b\}\uparrow$ we will write $b\uparrow$. Clearly, $A\uparrow = \bigcup_{a \in A} a\uparrow$ and we will say that a set A is **upward closed** iff $A = A\uparrow$. In a similar way we define $A\downarrow$, $b\downarrow$ and **downward closed** sets.

2 Topological preliminaries

A **sequence** in a set X is each function $x : \omega \to X$; instead of $x(n)$ we usually write x_n and also $x = \langle x_n : n \in \omega \rangle$. The **constant sequence** $\langle a, a, a, \ldots \rangle$ is denoted by $\langle a \rangle$. If $f \in \omega^{\uparrow\omega}$, the sequence $y = x \circ f$ is said to be a **subsequence** of the sequence x and we write $y \prec x$.

If $\langle X, \mathcal{O} \rangle$ is a topological space, a point $a \in X$ is said to be a **limit point of a sequence** $x \in X^\omega$ (we will write: $x \to_{\mathcal{O}} a$) iff each neighborhood U of a contains all but finitely many members of the sequence. A space $\langle X, \mathcal{O} \rangle$ is called **sequential** iff a set $A \subset X$ is closed whenever it contains each limit of each sequence in A.

If X is a non-empty set, each mapping $\lambda : X^\omega \to P(X)$ is a **convergence** on X and the mapping $u_\lambda : P(X) \to P(X)$, defined by $u_\lambda(A) = \bigcup_{x \in A^\omega} \lambda(x)$, the **operator of sequential closure** determined by λ. A convergence λ satisfying $|\lambda(x)| \leq 1$, for each sequence x in X, is called a **Hausdorff convergence**. If λ_1 is another convergence on X, then we will write $\lambda \leq \lambda_1$ iff $\lambda(x) \subset \lambda_1(x)$, for each sequence $x \in X^\omega$. Clearly, $\leq$ is a partial ordering on the set $\mathrm{Conv}(X) = \{\lambda : \lambda \text{ is a convergence on } X\}$.

If $\langle X, \mathcal{O} \rangle$ is a topological space, then the mapping $\lim_{\mathcal{O}} : X^\omega \to P(X)$ defined by $\lim_{\mathcal{O}}(x) = \{a \in X : x \to_{\mathcal{O}} a\}$ is **the convergence on** X **determined by the topology** $\mathcal{O}$ and for the operator $\lambda = \lim_{\mathcal{O}}$ we have (see [6])

(L1) $\forall a \in X\ \ a \in \lambda(\langle a \rangle)$;
(L2) $\forall x \in X^\omega\ \forall y \prec x\ \lambda(x) \subset \lambda(y)$;

(L3) $\forall x \in X^\omega \; \forall a \in X \; ((\forall y \prec x \; \exists z \prec y \; a \in \lambda(z)) \Rightarrow a \in \lambda(x))$.

We will use the following facts which mainly belong to the topological folklore. Their proofs can be found in [11].

Fact 2.1 If $\mathcal{O}_1$ and $\mathcal{O}_2$ are topologies on a set X, then
(a) $\mathcal{O}_1 \subset \mathcal{O}_2$ implies $\lim_{\mathcal{O}_2} \leq \lim_{\mathcal{O}_1}$.
(b) If $\mathcal{O}_1$ and $\mathcal{O}_2$ are sequential topologies and $\lim_{\mathcal{O}_1} = \lim_{\mathcal{O}_2}$, then $\mathcal{O}_1 = \mathcal{O}_2$.

A convergence $\lambda : X^\omega \to P(X)$ is called a **topological convergence** iff there is a topology $\mathcal{O}$ on X such that $\lambda = \lim_{\mathcal{O}}$. The following fact shows that each convergence has a minimal topological extension and connects topological and convergence structures.

Fact 2.2 Let $\lambda : X^\omega \to P(X)$ be a convergence on a non-empty set X. Then
(a) There is the maximal topology $\mathcal{O}_\lambda$ on X satisfying $\lambda \leq \lim_{\mathcal{O}_\lambda}$;
(b) $\mathcal{O}_\lambda = \{O \subset X : \forall x \in X^\omega \, (O \cap \lambda(x) \neq \emptyset \Rightarrow \exists n_0 \in \omega \, \forall n \geq n_0 \, x_n \in O)\}$;
(c) $\langle X, \mathcal{O}_\lambda \rangle$ is a sequential space;
(d) $\mathcal{O}_\lambda = \{X \setminus F : F \subset X \wedge u_\lambda(F) = F\}$, if λ satisfies (L1) and (L2);
(e) $\lim_{\mathcal{O}_\lambda} = \min\{\lambda' \in \mathrm{Conv}(X) : \lambda'$ is topological and $\lambda \leq \lambda'\}$;
(f) $\mathcal{O}_{\lim_{\mathcal{O}_\lambda}} = \mathcal{O}_\lambda$;
(g) If $\lambda_1 : X^\omega \to P(X)$ and $\lambda_1 \leq \lambda$, then $\mathcal{O}_\lambda \subset \mathcal{O}_{\lambda_1}$;
(h) λ is a topological convergence iff $\lambda = \lim_{\mathcal{O}_\lambda}$.

If a convergence λ satisfies conditions (L1) and (L2), then the minimal closure of λ under (L3) is described in the following statement.

Fact 2.3 Let $\lambda : X^\omega \to P(X)$ be a convergence satisfying (L1) and (L2). Then the mapping $\lambda^* : X^\omega \to P(X)$ given by $\lambda^*(y) = \bigcap_{f \in \omega^{\uparrow\omega}} \bigcup_{g \in \omega^{\uparrow\omega}} \lambda(y \circ f \circ g)$ is the minimal convergence bigger than λ and satisfying (L1) - (L3). Hence $\lambda^* \leq \lim_{\mathcal{O}_\lambda}$.

A convergence $\lambda : X^\omega \to P(X)$ will be called **weakly-topological** iff it satisfies conditions (L1) and (L2) and its (L3)-closure, λ^*, is a topological convergence.

Fact 2.4 Let $\lambda : X^\omega \to P(X)$ be a convergence satisfying (L1) and (L2).
(a) λ is weakly topological iff $\lambda^* = \lim_{\mathcal{O}_\lambda}$, that is iff for each $x \in X^\omega$ and $a \in X$ we have: $a \in \lim_{\mathcal{O}_\lambda}(x) \Leftrightarrow \forall y \prec x \;\; \exists z \prec y \;\; a \in \lambda(z)$;
(b) If λ is a Hausdorff convergence, then λ^* is also a Hausdorff convergence and $\lambda^* = \lim_{\mathcal{O}_\lambda}$, that is λ is a weakly-topological convergence.

Fact 2.5 Let $\lambda : X^\omega \to P(X)$ be a convergence satisfying (L1) and (L2) and let the mappings $u^\alpha_\lambda : P(X) \to P(X)$, $\alpha \leq \omega_1$, be defined by recursion in the following way: for $A \subset X$

$u_\lambda^0(A) = A$,
$u_\lambda^{\alpha+1}(A) = u_\lambda(u_\lambda^\alpha(A))$ and
$u_\lambda^\gamma(A) = \bigcup_{\alpha<\gamma} u_\lambda^\alpha(A)$, for a limit $\gamma \leq \omega_1$.
Then $u_\lambda^{\omega_1}$ is the closure operator in the space $\langle X, \mathcal{O}_\lambda \rangle$.

3 The Cantor cube and the algebraic convergence

First we recall that if X_n, $n \in \omega$, is a sequence of sets, then $\liminf_{n\in\omega} X_n = \bigcup_{k\in\omega} \bigcap_{n\geq k} X_n = \{x : x \in X_n$ for all but finitely many $n\}$ and $\limsup_{n\in\omega} X_n = \bigcap_{k\in\omega} \bigcup_{n\geq k} X_n = \{x : x \in X_n$ for infinitely many $n\}$. Clearly we have

Fact 3.1 Let X_n, $n \in \omega$, be a sequence of sets. Then
(a) $\liminf_{n\in\omega} X_n \subset \limsup_{n\in\omega} X_n$;
(b) If $X_n = X$, for each $n \geq k$, then $\liminf_{n\in\omega} X_n = \limsup_{n\in\omega} X_n = X$.

We remind the reader that if κ is an infinite cardinal, then the Cantor cube of weight κ, denoted by $\langle 2^\kappa, \tau_{\mathrm{C}} \rangle$, is the Tychonov product of κ many copies of the two point discrete space $2 = \{0, 1\}$. We will identify the set 2^κ with the power set $P(\kappa)$ using the bijection $f : 2^\kappa \to P(\kappa)$ defined by $f(x) = x^{-1}[\{1\}]$.

Fact 3.2 Let $\langle x_n \rangle$ be a sequence in 2^κ and $x \in 2^\kappa$. Then the following conditions are equivalent:
(a) $\langle x_n \rangle \to_{\tau_{\mathrm{C}}} x$,
(b) $\forall \alpha \in \kappa \ \exists k \in \omega \ \forall n \geq k \ x_n(\alpha) = x(\alpha)$,
(c) $\liminf_{n\in\omega} X_n = \limsup_{n\in\omega} X_n = X$, where $X_n = f(x_n)$ and $X = f(x)$.
The Cantor cube $\langle 2^\kappa, \tau_{\mathrm{C}} \rangle$ is a sequential space iff $\kappa = \omega$.

Proof. (a) $\Leftrightarrow$ (b) is true since the topology on the set 2 is discrete and the convergence of sequences in Tychonov products is the pointwise convergence (see [6]).

(b) $\Rightarrow$ (c). By (b), for each $\alpha \in \kappa$ there is $k \in \omega$ such that $X_n \cap \{\alpha\} = X \cap \{\alpha\}$, for each $n \geq k$, which, by Fact 3.1(b), implies that $\liminf_{n\in\omega} X_n \cap \{\alpha\} = \limsup_{n\in\omega} X_n \cap \{\alpha\} = X \cap \{\alpha\}$. This holds for all $\alpha \in \kappa$ so (c) is true.

(c) $\Rightarrow$ (b). Assuming (c), in order to prove (b) we take $\alpha \in \kappa$. If $\alpha \in X$ then, by (c), there is $k \in \omega$ such that for each $n \geq k$ we have $\alpha \in X_n$, that is $x_n(\alpha) = 1 = x(\alpha)$. If $\alpha \in \kappa \setminus X = \bigcup_{k\in\omega} \bigcap_{n\geq k} \kappa \setminus X_n$ then there is $k \in \omega$ such that for each $n \geq k$ we have $\alpha \in \kappa \setminus X_n$, that is $x_n(\alpha) = 0 = x(\alpha)$ and (b) is true.

The Cantor space 2^ω is sequential, since it is metrizable (see [6]).

Let $\kappa > \omega$ and let $A \subset 2^\kappa$ be the family of characteristic functions of at most countable subsets of κ. By (a) and since the limit superior of a sequence of

countable sets is countable, the set A is sequentially closed. But A is dense in 2^κ and, hence, not closed. Thus $\langle 2^\kappa, \tau_{\mathrm{C}}\rangle$ is not a sequential space. □

Let the convergence λ_{s} on the power set $P(\kappa)$ be defined by

$$\lambda_{\mathrm{s}}(\langle X_n\rangle) = \begin{cases} \{X\} & \text{if } \liminf X_n = \limsup X_n = X, \\ \emptyset & \text{if } \liminf X_n < \limsup X_n. \end{cases}$$

Fact 3.3 Let $f : 2^\kappa \to P(\kappa)$ be the bijection given by $f(x) = x^{-1}[\{1\}]$. Then
(a) $\tau_{\mathrm{C}}^{P(\kappa)} = \{f[O] : O \in \tau_{\mathrm{C}}\}$ is a topology on the power set algebra $P(\kappa)$;
(b) $f : \langle 2^\kappa, \tau_{\mathrm{C}}\rangle \to \langle P(\kappa), \tau_{\mathrm{C}}^{P(\kappa)}\rangle$ is a homeomorphism;
(c) $\lambda_{\mathrm{s}} = \lim_{\tau_{\mathrm{C}}^{P(\kappa)}} = \lim_{\mathcal{O}_{\lambda_{\mathrm{s}}}}$, thus λ_{s} is a topological convergence;
(d) $\mathcal{O}_{\lambda_{\mathrm{s}}} = \tau_{\mathrm{C}}^{P(\kappa)}$ iff $\kappa = \omega$. If $\kappa > \omega$, then $\tau_{\mathrm{C}}^{P(\kappa)} \subsetneq \mathcal{O}_{\lambda_{\mathrm{s}}}$.

Proof. (a) and (b) are evident. Let us prove (c). By Fact 3.2, $X \in \lambda_{\mathrm{s}}(\langle X_n\rangle)$ iff $\langle x_n\rangle \to_{\tau_{\mathrm{C}}} x$ which is, by (b), equivalent to $X \in \lim_{\tau_{\mathrm{C}}^{P(\kappa)}}(\langle X_n\rangle)$. Now, the second equality follows from Fact 2.2(h). (d) follows from Fact 3.2 and Fact 2.2(c). □

The convergence λ_{s} on the power set algebras is generalized for an arbitrary complete Boolean algebra $\mathbb{B}$ defining the **algebraic convergence** λ_{s} on $\mathbb{B}$ by

$$\lambda_{\mathrm{s}}(\langle x_n\rangle) = \begin{cases} \{x\} & \text{if } \liminf x_n = \limsup x_n = x, \\ 0 & \text{if } \liminf x_n < \limsup x_n, \end{cases}$$

where $\liminf x_n = \bigvee_{k\in\omega} \bigwedge_{n\geq k} x_n$ and $\limsup x_n = \bigwedge_{k\in\omega} \bigvee_{n\geq k} x_n$. By Fact 2.2, there is the maximal topology $\mathcal{O}_{\lambda_{\mathrm{s}}}$ on $\mathbb{B}$ such that $\lambda_{\mathrm{s}} \leq \lim_{\mathcal{O}_{\lambda_{\mathrm{s}}}}$, called the **sequential topology**, traditionally denoted by τ_{s}. It played a significant role in the solution of von Neumann's [15] and Maharam's Problem [14] solved by Talagrand [16, 17] (see also papers of Balcar, Glówczyński and Jech [1]; Balcar, Jech and Pazák [2]; Balcar and Jech [3]; Farah [7]; Todorčević [18] and Veličković [19]).

It is known that the convergence λ_{s} is weakly-topological. Namely we have

Fact 3.4 Let $\mathbb{B}$ be a complete Boolean algebra. Then
(a) λ_{s} is a Hausdorff convergence satisfying (L1) and (L2);
(b) λ_{s} is a weakly-topological convergence.

Proof. Clearly, λ_{s} is a Hausdorff convergence and satisfies (L1). Since for each $x, y \in \mathbb{B}$, $y \prec x$ implies $\liminf x \leq \liminf y \leq \limsup y \leq \limsup x$, it satisfies (L2). (b) follows from (a) and Fact 2.4. □

By Fact 3.3(c), on each power set algebra the convergence λ_{s} is topological. In general, by Fact 2.2(h) and Theorem 2 of [10] we have

Theorem 3.5 For each c.B.a. $\mathbb{B}$ the following conditions are equivalent:
(a) $\lambda_{\rm s}$ is a topological convergence;
(b) $\lambda_{\rm s} = \lim_{\mathcal{O}_{\lambda_{\rm s}}}$;
(c) The algebra $\mathbb{B}$ is $(\omega, 2)$-distributive;
(d) Forcing by $\mathbb{B}$ does not produce new reals.

If an algebra $\mathbb{B}$ is not $(\omega, 2)$-distributive but

$$\forall x \in \mathbb{B}^\omega \ \exists y \prec x \ \forall z \prec y \ \limsup z = \limsup y, \qquad (\hbar)$$

then the convergence $\lim_{\mathcal{O}_{\lambda_{\rm s}}}$ is characterized in the following way (see [10]).

Theorem 3.6 If a complete Boolean algebra $\mathbb{B}$ satisfies condition $(\hbar)$, then for each sequence $x \in \mathbb{B}^\omega$ and $a \in \mathbb{B}$ we have: $a \in \lim_{\mathcal{O}_{\lambda_{\rm s}}}(x) \Leftrightarrow a_x = b_x = a$, where

$$a_x = \bigwedge_{A\in[\omega]^\omega} \bigvee_{B\in[A]^\omega} \bigwedge_{n\in B} x_n \quad \text{and} \quad b_x = \bigvee_{A\in[\omega]^\omega} \bigwedge_{B\in[A]^\omega} \bigvee_{n\in B} x_n.$$

The implication "$\Rightarrow$" holds in each c.B.a.

We note that, by [10], condition ($\hbar$) is related to the cellularity of complete Boolean algebras: $\mathfrak{t}$-cc $\Rightarrow (\hbar) \Rightarrow \mathfrak{s}$-cc. By [13], $\{\kappa \in \mathrm{Card} : \kappa\text{-cc} \Rightarrow (\hbar)\}$ is either $[0, \mathfrak{h})$ or $[0, \mathfrak{h}]$ and $\{\kappa \in \mathrm{Card} : (\hbar) \Rightarrow \kappa\text{-cc}\} = [\mathfrak{s}, \infty)$.

4 The Aleksandrov cube and the convergences $\lambda_{\rm ls}$ and $\lambda_{\rm li}$

We remind the reader that the Aleksandrov cube of weight κ, here denoted by $\langle 2^\kappa, \tau_A\rangle$, is the Tychonov product of κ-many copies of the two-point space $2 = \{0, 1\}$ with the topology $\mathcal{O}_A = \{\emptyset, \{0\}, \{0, 1\}\}$. It is an universal T_0 space of weight κ (see [6]).

Fact 4.1 (a) Let $\langle x_n\rangle$ be a sequence in 2^κ and $x \in 2^\kappa$. Then $\langle x_n\rangle \to_{\tau_A} x$ iff

$$\limsup_{n\in\omega} X_n \subset X, \qquad (1)$$

where $X_n = x_n^{-1}[\{1\}]$, for $n \in \omega$, and $X = x^{-1}[\{1\}]$.
(b) $\langle 2^\kappa, \tau_A\rangle$ is a sequential space iff $\kappa = \omega$.

Proof. (a) In the space $\langle 2, \mathcal{O}_A\rangle$ the point 0 is isolated and the only neighborhood of the point 1 is $\{0, 1\}$ so, a sequence $\langle a_n : n \in \omega\rangle$ converges to a point a iff $a = 1$, or $a = 0$ and there is $k \in \omega$ such that $a_n = 0$, for all $n \geq k$. Now as in Section 3 we conclude that, in the space $\langle 2^\kappa, \tau_A\rangle$, $\langle x_n\rangle \to_{\tau_A} x$ iff for each $\alpha < \kappa$, $\langle x_n(\alpha)\rangle \to_{\mathcal{O}_A} x(\alpha)$ iff

$$\forall \alpha < \kappa \ \Big[x(\alpha) = 1 \ \vee \ \Big(x(\alpha) = 0 \ \wedge \ \exists k \in \omega \ \ \forall n \geq k \ \ x_n(\alpha) = 0\Big)\Big]$$

iff for each $\alpha < \kappa$ we have $\alpha \in X \vee \neg\forall k \in \omega\ \exists n \geq k\ \ \alpha \in X_n$, that is $\alpha \in \limsup X_n \Rightarrow \alpha \in X$.

(b) ($\Leftarrow$) $\langle 2^\omega, \tau_A\rangle$ is a first countable and, consequently, a sequential space.

($\Rightarrow$) Let $\kappa > \omega$. The set $S = \{x \in 2^\kappa : |x^{-1}[\{0\}]| \leq \aleph_0\}$ is dense in the space $\langle 2^\kappa, \tau_A\rangle$ and, hence, it is not closed. In order to show that S is sequentially closed we take a sequence $\langle x_n : n \in \omega\rangle$ in S and show that $\lim_{\tau_A}(\langle x_n\rangle) \subset S$. The corresponding sets $X_n = x_n^{-1}[\{1\}]$, $n \in \omega$, are co-countable subsets of κ, thus $X_n = \kappa \setminus C_n$, where $C_n \in [\kappa]^{\leq\aleph_0}$ and the set $\limsup X_n = \kappa \setminus \bigcup_{k\in\omega}\bigcap_{n\geq k} C_n$ is co-countable as well. By (a), if $x \in \lim_{\tau_A}(\langle x_n\rangle)$, then $\limsup X_n \subset X$, which means that X is a co-countable set and, consequently, $x \in S$. $\Box$

Let the convergence λ_{ls} on $P(\kappa)$ be defined by

$$\lambda_{\mathrm{ls}}(\langle X_n\rangle) = (\limsup X_n)\uparrow .$$

Theorem 4.2 Let $f : 2^\kappa \to P(\kappa)$ be the bijection given by $f(x) = x^{-1}[\{1\}]$. Then

(a) $\tau_{\mathrm{A}}^{P(\kappa)} = \{f[O] : O \in \tau_{\mathrm{A}}\}$ is a topology on $P(\kappa)$;

(b) $f : \langle 2^\kappa, \tau_{\mathrm{A}}\rangle \to \langle P(\kappa), \tau_{\mathrm{A}}^{P(\kappa)}\rangle$ is a homeomorphism;

(c) $\lambda_{\mathrm{ls}} = \lim_{\tau_{\mathrm{A}}^{P(\kappa)}} = \lim_{\mathcal{O}_{\lambda_{\mathrm{ls}}}}$ and λ_{ls} is a topological convergence.

(d) $\mathcal{O}_{\lambda_{\mathrm{ls}}} = \tau_{\mathrm{A}}^{P(\kappa)}$ iff $\kappa = \omega$. If $\kappa > \omega$, then $\tau_{\mathrm{A}}^{P(\kappa)} \subsetneq \mathcal{O}_{\lambda_{\mathrm{ls}}}$.

(e) $\mathcal{O}_{\lambda_{\mathrm{ls}}} \not\subset \tau_{\mathrm{C}}^{P(\kappa)}$, if $\kappa > \omega$.

Proof. (a) and (b) are evident. (c) and (d) follow from Fact 4.1 and Fact 2.2(h).

(e) As in Fact 4.1 we consider the set $F = \{\kappa \setminus C : C \in [\kappa]^{\leq\aleph_0}\}$, which is dense in the space $\langle P(\kappa), \tau_{\mathrm{C}}^{P(\kappa)}\rangle$ and, hence $P(\kappa) \setminus F \notin \tau_{\mathrm{C}}^{P(\kappa)}$. If $\langle X_n\rangle$ is a sequence in F, where $X_n = \kappa \setminus C_n$, then $\limsup X_n = \kappa \setminus \bigcup_{k\in\omega}\bigcap_{n\geq k} C_n \in F$ and, clearly, $\lambda_{\mathrm{ls}}(\langle X_n\rangle) = (\limsup X_n)\uparrow \subset F$, thus $u_{\lambda_{\mathrm{ls}}}(F) = F$. By (c), λ_{ls} satisfies (L1) and (L2) so, by Fact 2.2(d), $P(\kappa) \setminus F \in \mathcal{O}_{\lambda_{\mathrm{ls}}}$. $\Box$

Now we generalize this for an arbitrary complete Boolean algebra $\mathbb{B}$ defining the convergence λ_{ls} by

$$\lambda_{\mathrm{ls}}(\langle x_n\rangle) = (\limsup x_n)\uparrow$$

and Fact 2.2 provides the topology $\mathcal{O}_{\lambda_{\mathrm{ls}}}$ on $\mathbb{B}$. We will also consider the dual convergence λ_{li} on $\mathbb{B}$ defined by $\lambda_{\mathrm{li}}(\langle x_n\rangle) = (\liminf x_n)\downarrow$ and the corresponding topology $\mathcal{O}_{\lambda_{\mathrm{li}}}$.

If λ_1 and λ_2 are convergences, by $\lambda_1 \cap \lambda_2$ we will denote the convergence defined by $(\lambda_1 \cap \lambda_2)(x) = \lambda_1(x) \cap \lambda_2(x)$. Similarly to Fact 3.4 we have

Theorem 4.3 Let $\mathbb{B}$ be a complete Boolean algebra. Then
(a) $\lambda_{\rm ls}$ and $\lambda_{\rm li}$ are non-Hausdorff convergences satisfying (L1) and (L2);
(b) $\lambda_{\rm s} = \lambda_{\rm ls} \cap \lambda_{\rm li}$ and, consequently, $\lambda_{\rm s} \leq \lambda_{\rm ls}, \lambda_{\rm li}$;
(c) $\mathcal{O}_{\lambda_{\rm ls}}, \mathcal{O}_{\lambda_{\rm li}} \subset \mathcal{O}_{\lambda_{\rm s}}$;
(d) $\lambda^*_{\rm ls} \leq \lim_{\mathcal{O}_{\lambda_{\rm ls}}}$ and $\lambda^*_{\rm li} \leq \lim_{\mathcal{O}_{\lambda_{\rm li}}}$.
(e) $\lambda^*_{\rm s} = \lambda^*_{\rm ls} \cap \lambda^*_{\rm li}$ and, consequently, $\lambda^*_{\rm s} \leq \lambda^*_{\rm ls}, \lambda^*_{\rm li}$.

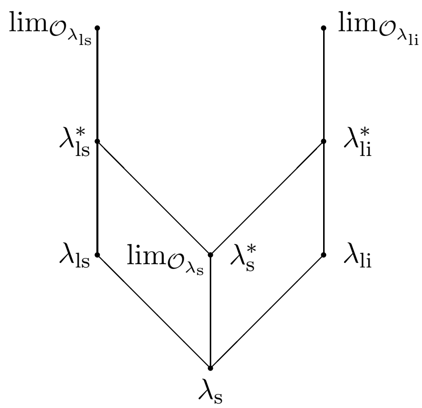
Proof. (a) Since $a \in a\uparrow = (\limsup\langle a \rangle)\uparrow = \lambda_{\rm ls}(\langle a \rangle)$, for each $a \in \mathbb{B}$, $\lambda_{\rm ls}$ satisfies (L1) and it is not Hausdorff because $0, 1 \in \lambda_{\rm ls}(\langle 0 \rangle)$. For a proof of (L2) note that $y \prec x$ implies $\limsup y \leq \limsup x$ so we have $(\limsup x)\uparrow \subset (\limsup y)\uparrow$, that is $\lambda_{\rm ls}(x) \subset \lambda_{\rm ls}(y)$.

(b) If $a \in \lambda_{\rm s}(x)$, then $a = \limsup x \in (\limsup x)\uparrow = \lambda_{\rm ls}(x)$ and, similarly, $a \in \lambda_{\rm li}(x)$. Conversely, if $a \in \lambda_{\rm ls}(x) \cap \lambda_{\rm li}(x)$, then $\limsup x \leq a \leq \liminf x$, which implies $\limsup x = \liminf x = a$, that is $a \in \lambda_{\rm s}(x)$.

(c) follows from (b) and Fact 2.2(g). (d) follows from Fact 2.3.

(e) By (b) and by the minimality of λ^* (see Fact 2.3) we have $\lambda^*_{\rm s} \leq \lambda^*_{\rm ls}, \lambda^*_{\rm li}$. So, it remains to be proved that $\lambda^*_{\rm ls} \cap \lambda^*_{\rm li} \leq \lambda^*_{\rm s}$. Let $x \in \mathbb{B}^\omega$ and $a \in \lambda^*_{\rm ls}(x) \cap \lambda^*_{\rm li}(x)$. If $y \prec x$, then there is $z \prec y$ such that $a \geq \limsup z$ and there is $t \prec z$ such that $a \leq \liminf t$. But then $\limsup t \leq \limsup z \leq a \leq \liminf t$, which implies $a \in \lambda_{\rm s}(x)$. Thus for each $y \prec x$ there is $t \prec y$ such that $a \in \lambda_{\rm s}(x)$, that is $a \in \lambda^*_{\rm s}(x)$. $\square$

By the previous theorem and Fact 3.4, the relations between the convergences considered in this paper are presented in the following diagram.



In the sequel we will use the following characterization, where the families of closed sets corresponding to the topologies $\mathcal{O}_{\lambda_{\rm ls}}$ and $\mathcal{O}_{\lambda_{\rm li}}$ are denoted by $\mathcal{F}_{\lambda_{\rm ls}}$ and $\mathcal{F}_{\lambda_{\rm li}}$ respectively.

Theorem 4.4 Let $\mathbb{B}$ be a complete Boolean algebra. Then
(I) For a set $F \subset \mathbb{B}$ the following conditions are equivalent:
(a) $F \in \mathcal{F}_{\lambda_{\mathrm{ls}}}$;
(b) F is upward closed and $\limsup x \in F$, for each sequence $x \in F^\omega$;
(c) F is upward closed and $\bigwedge_{n\in\omega} x_n \in F$, for each decreasing $x \in F^\omega$.
(II) For a set $F \subset \mathbb{B}$ the following conditions are equivalent:
(a) $F \in \mathcal{F}_{\lambda_{\mathrm{li}}}$;
(b) F is downward closed and $\liminf x \in F$, for each sequence $x \in F^\omega$;
(c) F is downward closed and $\bigvee_{n\in\omega} x_n \in F$, for each increasing $x \in F^\omega$.
(III) The mapping $h : \langle \mathbb{B}, \mathcal{O}_{\lambda_{\mathrm{ls}}}\rangle \to \langle \mathbb{B}, \mathcal{O}_{\lambda_{\mathrm{li}}}\rangle$ given by $h(b) = b'$, for each $b \in \mathbb{B}$, is a homeomorphism.

Proof. We prove (I). The proof of (II) is dual.

(a) $\Rightarrow$ (b). Let $X \setminus F \in \mathcal{O}_{\lambda_{\mathrm{ls}}}$. Then, by Theorems 4.3 and Fact 2.2(d) we have $F = u_{\lambda_{\mathrm{ls}}}(F) = \bigcup_{x\in F^\omega}(\limsup x)\uparrow$ and, hence, F is upward closed. Also, if $x \in F^\omega$, then $\limsup x \in (\limsup x)\uparrow \subset F$.

(b) $\Rightarrow$ (c). If $x \in F^\omega$ is a decreasing sequence, $\bigwedge_{n\in\omega} x_n = \limsup x \in F$.

(c) $\Rightarrow$ (a). Assuming (c), by Fact 2.2(d) we show that $u_{\lambda_{\mathrm{ls}}}(F) = F$. If $b \in u_{\lambda_{\mathrm{ls}}}(F)$, then there is $x \in F^\omega$ such that $b \geq \limsup x$. Since the set F is upward closed and $x_n \in F$, for $k \in \omega$ we have $y_k = b \vee \bigvee_{n\geq k} x_n \in F$ and, clearly, $y = \langle y_k : k \in \omega\rangle$ is a decreasing sequence. So, $F \ni \bigwedge_{k\in\omega} y_k = \bigwedge_{k\in\omega}(b \vee \bigvee_{n\geq k} x_n) = b \vee \bigwedge_{k\in\omega}\bigvee_{n\geq k} x_n = b \vee \limsup x = b$.

(III) h is a bijection and for a proof of its continuity we take $F \in \mathcal{F}_{\lambda_{\mathrm{li}}}$ and show that $h^{-1}[F] = \{b' : b \in F\} \in \mathcal{F}_{\lambda_{\mathrm{ls}}}$. If $a \geq b' \in h^{-1}[F]$, then $a' \leq b \in F$ and, by (II), $a' \in F$, which implies $a \in h^{-1}[F]$. Thus the set $h^{-1}[F]$ is upward closed. Let $\langle x_n\rangle$ be a decreasing sequence in $h^{-1}[F]$. Then $\langle x_n'\rangle$ is an increasing sequence in F and, by (II) again, $\bigvee_{n\in\omega} x_n' = (\bigwedge_{n\in\omega} x_n)' \in F$, which implies $\bigwedge_{n\in\omega} x_n \in h^{-1}[F]$. By (I), $h^{-1}[F] \in \mathcal{F}_{\lambda_{\mathrm{ls}}}$. The proof that h is closed is similar. $\Box$

5 The algebras with λ_{ls} topological

In this section we prove the following characterization of complete Boolean algebras on which the convergences λ_{ls} and λ_{li} are topological.

Theorem 5.1 For each complete Boolean algebra $\mathbb{B}$ the following conditions are equivalent:
(a) λ_{ls} is a topological convergence;
(b) λ_{li} is a topological convergence;
(c) $\mathbb{B}$ is an $(\omega, 2)$-distributive algebra;
(d) Forcing by $\mathbb{B}$ does not produce new reals.

The following three lemmas will be used in our proof.

Lemma 5.2 Let $\mathbb{B}$ be a complete Boolean algebra. Then

(a) For each $a \in \mathbb{B}$ the function $f_a : \langle \mathbb{B}, \mathcal{O}_{\lambda_{\rm ls}} \rangle \to \langle \mathbb{B}, \mathcal{O}_{\lambda_{\rm ls}} \rangle$ defined by $f_a(x) = x \wedge a$ is continuous;

(b) $\lim_{\mathcal{O}_{\lambda_{\rm ls}}} \neq \lambda_{\rm ls}$ iff there is a sequence x in $\mathbb{B}$ such that $0 \in \lim_{\mathcal{O}_{\lambda_{\rm ls}}}(x) \setminus \lambda_{\rm ls}(x)$.

(c) If $x, y \in \mathbb{B}^\omega$ and $x_n \leq y_n$, for each $n \in \omega$, then $\lim_{\mathcal{O}_{\lambda_{\rm ls}}}(y) \subset \lim_{\mathcal{O}_{\lambda_{\rm ls}}}(x)$.

Proof. (a) By Theorem 4.4 we show that for a closed set $F \subset \mathbb{B}$ the set $f_a^{-1}[F] = \{x \in \mathbb{B} : x \wedge a \in F\}$ is upward closed and contains the infimum of each decreasing sequence in $f_a^{-1}[F]$. First, if $x_1 \geq x \in f_a^{-1}[F]$, then $x_1 \wedge a \geq x \wedge a \in F$ and, since F is upward closed, $x_1 \wedge a \in F$, that is $x_1 \in f_a^{-1}[F]$. Second, if $\langle x_n \rangle$ is a decreasing sequence in $f_a^{-1}[F]$, then $\langle x_n \wedge a \rangle$ is a decreasing sequence in F and, since F is closed, $\bigwedge_{n \in \omega} x_n \wedge a \in F$, thus $\bigwedge_{n \in \omega} x_n \in f_a^{-1}[F]$.

(b) Let $y \in \mathbb{B}^\omega$ and $b \in \lim_{\mathcal{O}_{\lambda_{\rm ls}}}(y) \setminus \lambda_{\rm ls}(y)$. Then $\limsup y \not\leq b$ and, hence, $c = \limsup y \wedge b' > 0$. Let $x = \langle y_n \wedge c : n \in \omega \rangle$. Since $c \leq \limsup y$ we have $c = \bigwedge_{k \in \omega} \bigvee_{n \geq k} y_n \wedge c = \limsup x$, which implies $0 \notin \lambda_{\rm ls}(x)$. Since $b \in \lim_{\mathcal{O}_{\lambda_{\rm ls}}}(y)$ and, by (a), the function $f_c : \mathbb{B} \to \mathbb{B}$ defined by $f_c(t) = t \wedge c$ is continuous, we have $0 = b \wedge c = f_c(b) \in \lim_{\mathcal{O}_{\lambda_{\rm ls}}}(\langle f_c(y_n) \rangle) = \lim_{\mathcal{O}_{\lambda_{\rm ls}}}(x)$.

(c) Let $a \in \lim_{\mathcal{O}_{\lambda_{\rm ls}}}(y)$ and $a \in O \in \mathcal{O}_{\lambda_{\rm ls}}$. Then there is $n_0 \in \omega$ such that for each $n \geq n_0$ we have $y_n \in O$, thus, since by Theorem 4.4 the set O is downward closed, $x_n \in O$, for $n \geq n_0$. So $a \in \lim_{\mathcal{O}_{\lambda_{\rm ls}}}(x)$. $\square$

If $x \in \mathbb{B}^\omega$, then $\tau_x = \{\langle \check{n}, x_n \rangle : n \in \omega\}$ is the corresponding $\mathbb{B}$-name for a subset of ω and, by Lemmas 2 and 6 of [10],

$$\begin{aligned}
\liminf x &= \|\check{\omega} \subset^* \tau_x\|; \\
\limsup x &= \||\tau_x| = \check{\omega}\|; \\
a_x &= \|\forall A \in (([\omega]^\omega)^V)\check{}\ \exists B \in (([A]^\omega)^V)\check{}\ B \subset^* \tau_x\|; \\
b_x &= \|\exists A \in (([\omega]^\omega)^V)\check{}\ \forall B \in (([A]^\omega)^V)\check{}\ |\tau_x \cap B| = \check{\omega}\|.
\end{aligned}$$

Lemma 5.3 Let $\mathbb{B}$ be a complete Boolean algebra and x a sequence in $\mathbb{B}$. Then

(a) $\liminf x \leq a_x \leq b_x \leq \limsup x$;

(b) If $\mathbb{B}$ is $(\omega, 2)$-distributive, then $a_x = \liminf x$ and $b_x = \limsup x$;

(c) $b_x = \bigvee_{y \prec x} \bigwedge_{z \prec y} \bigvee_{m \in \omega} z_m$.

Proof. (a) This is Lemma 7 of [10].

(b) Let $\mathbb{B}$ be $(\omega, 2)$-distributive. By (a), it is sufficient to show that $\limsup x \leq b_x$, that is $1 \Vdash |\tau_x| = \check{\omega} \Rightarrow \exists A \in (([\omega]^\omega)^V)\check{}\ \forall B \in (([A]^\omega)^V)\check{}\ |\tau_x \cap B| = \check{\omega}$. Let G be a $\mathbb{B}$-generic filter over V and let $|(\tau_x)_G| = \omega$. Then, by the $(\omega, 2)$-distributivity

we have $(\tau_x)_G \in ([\omega]^\omega)^V$ and $A = (\tau_x)_G$ is as required. Thus $b_x = \limsup x$. The proof of $a_x = \liminf x$ is similar.

(c) Clearly we have $\bigvee_{y\prec x}\bigwedge_{z\prec y}\bigvee_{m\in\omega} z_m = \bigvee_{f\in\omega^{\uparrow\omega}}\bigwedge_{z\prec x\circ f}\bigvee_{m\in\omega} z_m = \bigvee_{f\in\omega^{\uparrow\omega}}\bigwedge_{g\in\omega^{\uparrow\omega}}\bigvee_{m\in\omega} x_{f(g(m))} = \bigvee_{f\in\omega^{\uparrow\omega}}\bigwedge_{g\in\omega^{\uparrow\omega}}\bigvee_{n\in f[g[\omega]]} x_n$ and we prove that in each generic extension $V_{\mathbb{B}}[G]$ conditions

$$\exists A \in ([\omega]^\omega)^V\ \forall B \in ([A]^\omega)^V\ B \cap (\tau_x)_G \neq \emptyset \text{ and} \tag{2}$$

$$\exists f \in (\omega^{\uparrow\omega})^V\ \forall g \in (\omega^{\uparrow\omega})^V\ f[g[\omega]] \cap (\tau_x)_G \neq \emptyset \tag{3}$$

are equivalent. Let (2) hold and let f_A be the increasing enumeration of the set A. Then $f_A \in (\omega^{\uparrow\omega})^V$ and for any $g \in (\omega^{\uparrow\omega})^V$ we have $f_A[g[\omega]] \in ([A]^\omega)^V$ thus, by the assumption, $f_A[g[\omega]] \cap (\tau_x)_G \neq \emptyset$.

Let (3) hold. Then $A = f[\omega] \in ([\omega]^\omega)^V$ and, if $B \in ([A]^\omega)^V$, then $f^{-1}[B] \in ([\omega]^\omega)^V$ and $g_{f^{-1}[B]} \in (\omega^{\uparrow\omega})^V$, where $g_{f^{-1}[B]}$ is the increasing enumeration of the set $f^{-1}[B]$. By the assumption we have $f[g_{f^{-1}[B]}[\omega]] \cap (\tau_x)_G \neq \emptyset$ and, since $f[g_{f^{-1}[B]}[\omega]] = f[f^{-1}[B]] = B$ (because $B \subset f[\omega]$), we have $B \cap (\tau_x)_G \neq \emptyset$. $\square$

A sequence x in a c.B.a. $\mathbb{B}$ will be called **lim sup-stable** (**lim inf-stable**, respectively) iff $\limsup y = \limsup x$ ($\liminf y = \liminf x$ respectively), for each subsequence y of x.

Lemma 5.4 Let $x = \langle x_n : n \in \omega\rangle$ be a sequence in a c.B.a. $\mathbb{B}$.

(a) If x is a lim sup-stable sequence, then in the space $\langle \mathbb{B}, \mathcal{O}_{\lambda_{\mathrm{ls}}}\rangle$ we have

$$\overline{\{x_n : n \in \omega\}} = (\limsup x)\uparrow \cup \bigcup_{n\in\omega} x_n\uparrow; \tag{4}$$

(b) If x is a lim inf-stable sequence, then in the space $\langle \mathbb{B}, \mathcal{O}_{\lambda_{\mathrm{li}}}\rangle$ we have

$$\overline{\{x_n : n \in \omega\}} = (\liminf x)\downarrow \cup \bigcup_{n\in\omega} x_n\downarrow . \tag{5}$$

Proof. We prove (a) and the proof of (b) is dual. Let $X = \{x_n : n \in \omega\}$. First we prove that

$$u_{\lambda_{\mathrm{ls}}}(X) = (\limsup x)\uparrow \cup \bigcup_{n\in\omega} x_n\uparrow . \tag{6}$$

Since $(\limsup x)\uparrow = \lambda_{\mathrm{ls}}(\langle x_n : n \in \omega\rangle)$ and $x_n\uparrow = \lambda_{\mathrm{ls}}(\langle x_n, x_n, \ldots\rangle)$, for each $n \in \omega$, the inclusion "$\supset$" in (6) is proved. By Theorems 4.3(a) and Fact 2.5 we have $\overline{X} = u^{\omega_1}_{\lambda_{\mathrm{ls}}}(X) \supset u_{\lambda_{\mathrm{ls}}}(X)$ and the inclusion "$\supset$" in (4) is true as well.

In order to prove the inclusion "$\subset$" in (6) we take $y \in X^\omega$. If y has a constant subsequence, say $\langle x_n, x_n, \ldots\rangle$, then $x_n \leq \limsup y$ and, hence, $\lambda_{\mathrm{ls}}(y) = (\limsup y)\uparrow \subset x_n\uparrow$ and we are done. Otherwise, by Ramsey's Theorem, there is $H \in [\omega]^\omega$ such that $y \restriction H$ is an injection. Let the function $f : H \to \omega$ be defined by $f(k) = \min\{n \in \omega : y_k = x_n\}$. Then for different $k_1, k_2 \in H$

we have $x_{f(k_1)} = y_{k_1} \neq y_{k_2} = x_{f(k_2)}$ and, hence, $f(k_1) \neq f(k_2)$. Thus f is an injection so, by Ramsey's Theorem again and since ω is a well ordering, there is $H_1 \in [H]^\omega$ such that $f \upharpoonright H_1$ is an increasing function. Now we have $y \succ \langle y_k : k \in H_1\rangle = \langle x_{f(k)} : k \in H_1\rangle \prec x$ and, since x is a $\limsup$-stable sequence, $\limsup y \geq \limsup\langle y_k : k \in H_1\rangle = \limsup x$, which implies $\lambda_{\mathrm{ls}}(y) = (\limsup y)\uparrow\subset (\limsup x)\uparrow$ and (6) is proved.

Now, we prove that

$$u_{\lambda_{\mathrm{ls}}}(X) = u_{\lambda_{\mathrm{ls}}}(u_{\lambda_{\mathrm{ls}}}(X)). \tag{7}$$

The inclusion "$\subset$" holds, since λ_{ls} satisfies (L1). In order to prove "$\supset$" for $y \in u_{\lambda_{\mathrm{ls}}}(X)^\omega$ we show that $\lambda_{\mathrm{ls}}(y) \subset u_{\lambda_{\mathrm{ls}}}(X)$. By (6) we have

$$\forall k \in \omega \ \ (y_k \geq \limsup x \vee \exists n \in \omega \ \ y_k \geq x_n).$$

If there exists $G \in [\omega]^\omega$ such that $y_k \geq \limsup x$, for each $k \in G$, then $\limsup x \leq \limsup\langle y_k : k \in G\rangle \leq \limsup y$, which implies $\lambda_{\mathrm{ls}}(y) = (\limsup y)\uparrow\subset (\limsup x)\uparrow \subset u_{\lambda_{\mathrm{ls}}}(X)$.

Otherwise, there is $k_0 \in \omega$ such that for all $k \geq k_0$ there is $n \in \omega$ such that $y_k \geq x_n$. Let $f : \omega \setminus k_0 \to \omega$ be defined by $f(k) = \min\{n \in \omega : x_n \leq y_k\}$. Then $x_{f(k)} \leq y_k$, for $k \in \omega \setminus k_0$.

If there are $H_0 \in [\omega\setminus k_0]^\omega$ and $n \in \omega$ such that $f(k) = n$, for each $k \in H_0$, then $\limsup y \geq \limsup\langle y_k : k \in H_0\rangle \geq x_n$, which implies $\lambda_{\mathrm{ls}}(y) = (\limsup y)\uparrow\subset x_n\uparrow\subset u_{\lambda_{\mathrm{ls}}}(X)$. Otherwise, by Ramsey's Theorem, there is $H_1 \in [\omega \setminus k_0]^\omega$ such that $f \upharpoonright H_1$ is an injection and, by Ramsey's Theorem again, there exists $H_2 \in [H_1]^\omega$ such that $f \upharpoonright H_2$ is an increasing mapping. Now $\langle y_k : k \in H_2\rangle \prec y$, which implies

$$\limsup\langle y_k : k \in H_2\rangle \leq \limsup y \tag{8}$$

and $\langle x_{f(k)} : k \in H_2\rangle \prec x$, which, since x is a $\limsup$-stable sequence, implies

$$\limsup\langle x_{f(k)} : k \in H_2\rangle = \limsup x. \tag{9}$$

Since $x_{f(k)} \leq y_k$ we have $\limsup\langle x_{f(k)} : k \in H_2\rangle \leq \limsup\langle y_k : k \in H_2\rangle$ and, by (8) and (9), $\limsup x \leq \limsup y$ so $\lambda_{\mathrm{ls}}(y) \subset u_{\lambda_{\mathrm{ls}}}(X)$ again.

Since the convergence λ_{ls} satisfies (L1) and (L2), by Fact 2.5 we have $\overline{X} = u^{\omega_1}_{\lambda_{\mathrm{ls}}}(X)$ and (4) follows from (6) and (7). $\square$

Proof of Theorem 5.1 (c) $\Leftrightarrow$ (d) is a well known fact (see [8]).

(a) $\Leftrightarrow$ (b). Assuming that $\lambda_{\mathrm{ls}} = \lim_{\mathcal{O}_{\lambda_{\mathrm{ls}}}}$ we prove that $\lambda_{\mathrm{li}} = \lim_{\mathcal{O}_{\lambda_{\mathrm{li}}}}$, that is $\lim_{\mathcal{O}_{\lambda_{\mathrm{li}}}}(x) \subset \lambda_{\mathrm{li}}(x)$, for each sequence x in $\mathbb{B}$. So, if $a \in \lim_{\mathcal{O}_{\lambda_{\mathrm{li}}}}(x)$, then, by Theorem 4.4(III), we have $a' \in \lim_{\mathcal{O}_{\lambda_{\mathrm{ls}}}}(\langle x'_n\rangle) = \lambda_{\mathrm{ls}}(\langle x'_n\rangle)$, that is $a' \geq \limsup x'_n$,

which implies $a \leq \liminf x_n$ and, hence, $a \in \lambda_{\mathrm{li}}(x)$. The proof of the converse is similar.

(a) $\Rightarrow$ (c). If λ_{ls} is a topological convergence, then λ_{li} is topological as well. By Theorem 4.3(c) we have $\mathcal{O}_{\lambda_{\mathrm{ls}}}, \mathcal{O}_{\lambda_{\mathrm{li}}} \subset \mathcal{O}_{\lambda_{\mathrm{s}}}$, and, by Fact 2.1, $\lim_{\mathcal{O}_{\lambda_{\mathrm{s}}}} \leq \lim_{\mathcal{O}_{\lambda_{\mathrm{ls}}}}, \lim_{\mathcal{O}_{\lambda_{\mathrm{li}}}}$ so, since λ_{ls} and λ_{li} are topological, $\lim_{\mathcal{O}_{\lambda_{\mathrm{s}}}} \leq \lambda_{\mathrm{ls}}, \lambda_{\mathrm{li}}$, which, by Theorem 4.3(b) implies $\lim_{\mathcal{O}_{\lambda_{\mathrm{s}}}} \leq \lambda_{\mathrm{ls}} \cap \lambda_{\mathrm{li}} = \lambda_{\mathrm{s}} \leq \lim_{\mathcal{O}_{\lambda_{\mathrm{s}}}}$. So, $\lambda_{\mathrm{s}} = \lim_{\mathcal{O}_{\lambda_{\mathrm{s}}}}$, that is λ_{s} is a topological convergence and, by Theorem 3.5, the algebra $\mathbb{B}$ is $(\omega, 2)$-distributive.

(c) $\Rightarrow$ (a). Suppose that the algebra $\mathbb{B}$ is $(\omega, 2)$-distributive and that λ_{ls} is not a topological convergence. Then, by Lemma 5.2(b), there exists a sequence x in $\mathbb{B}$ such that $0 \in \lim_{\mathcal{O}_{\lambda_{\mathrm{ls}}}}(x)$ and $0 \not\in \lambda_{\mathrm{ls}}(x) = (\limsup x) \uparrow$, which implies $\limsup x = b > 0$. By Lemma 5.3 (b) and (c) we have $b_x = b$ and $\bigvee_{y \prec x} \bigwedge_{z \prec y} \bigvee_{n \in \omega} z_n = b$. Consequently, there exists $y \prec x$ and $c \in \mathbb{B}^+$ such that $\bigwedge_{z \prec y} \bigvee_{n \in \omega} z_n = c$, which implies

$$\forall z \prec y \ \bigvee_{n \in \omega} z_n \geq c. \tag{10}$$

Claim 1. $\langle y_n \wedge c : n \in \omega \rangle$ is a $\limsup$-stable sequence.

Proof of Claim 1. First, by (10) and since $\langle y_n : n \geq k \rangle$ is a subsequence of y, we have $\limsup \langle y_n \wedge c : n \in \omega \rangle = \bigwedge_{k \in \omega} (\bigvee_{n \geq k} y_n) \wedge c = \bigwedge_{k \in \omega} c = c$. Now we prove the same for an arbitrary subsequence $\langle y_{f(k)} \wedge c : k \in \omega \rangle$ of $\langle y_n \wedge c : n \in \omega \rangle$, where $f \in \omega^{\uparrow\omega}$. Clearly, $z = \langle y_{f(k)} : k \in \omega \rangle$ is a subsequence of y and for each $l \in \omega$ we have $\langle y_{f(k)} : k \geq l \rangle \prec y$, which, by (10), implies $\bigvee_{k \geq l} y_{f(k)} \geq c$. So, $\limsup \langle y_{f(k)} \wedge c : k \in \omega \rangle = \bigwedge_{l \in \omega} \bigvee_{k \geq l} y_{f(k)} \wedge c = \bigwedge_{l \in \omega} (\bigvee_{k \geq l} y_{f(k)}) \wedge c = \bigwedge_{l \in \omega} c = c$. Claim 1 is proved.

Claim 2. The set $M = \{n \in \omega : y_n \wedge c = 0\}$ is finite.

Proof of Claim 2. Suppose that $M \in [\omega]^\omega$. Then $\langle y_n \wedge c : n \in M \rangle$ is a subsequence of the sequence $\langle y_n \wedge c : n \in \omega \rangle$ and, clearly, $\limsup \langle y_n \wedge c : n \in M \rangle = 0 < c$, which is impossible by Claim 1. Claim 2 is proved.

By Claim 2, without loss of generality, we suppose that $y_n \wedge c > 0$, for each $n \in \omega$. By Theorem 5.4 we have $\overline{\{y_n \wedge c : n \in \omega\}} = c\uparrow \cup \bigcup_{n \in \omega} (y_n \wedge c) \uparrow$ and this set is closed in the space $\langle \mathbb{B}, \mathcal{O}_{\lambda_{\mathrm{ls}}} \rangle$, does not contain 0, but contains each element of the sequence $\langle y_n \wedge c : n \in \omega \rangle$. This implies $0 \not\in \lim_{\mathcal{O}_{\lambda_{\mathrm{ls}}}} \langle y_n \wedge c \rangle$.

On the other hand, since $y \prec x$ and $0 \in \lim_{\mathcal{O}_{\lambda_{\mathrm{ls}}}}(x)$, by (L2) we have $0 \in \lim_{\mathcal{O}_{\lambda_{\mathrm{ls}}}}(y)$. Since $y_n \wedge c \leq y_n$, for each $n \in \omega$, by Lemma 5.2(c) we have $\lim_{\mathcal{O}_{\lambda_{\mathrm{ls}}}}(y) \subset \lim_{\mathcal{O}_{\lambda_{\mathrm{ls}}}} \langle y_n \wedge c \rangle$ and, hence, $0 \in \lim_{\mathcal{O}_{\lambda_{\mathrm{ls}}}} \langle y_n \wedge c \rangle$. A contradiction. $\square$

6 The algebras with λ_{ls} weakly topological

By Theorem 5.1, if a complete Boolean algebra is not $(\omega, 2)$-distributive, the convergences λ_{ls} and λ_{li} are not topological. Now we show that they are weakly topological in algebras satisfying condition ($\hbar$). The reader will notice that if in condition ($\hbar$) we replace "$\limsup$" by "$\liminf$", then we obtain an equivalent condition, because $(\limsup x_n)' = \liminf x'_n$, for each sequence x in $\mathbb{B}$.

Theorem 6.1 If $\mathbb{B}$ is a complete Boolean algebra satisfying condition $(\hbar)$, then λ_{ls} and λ_{li} are weakly topological convergences.

Proof. We prove the statement for λ_{ls}. The proof for λ_{li} is dual. We show that for each sequence x in $\mathbb{B}$ and each $a \in \mathbb{B}$ we have $a \in \lim_{\mathcal{O}_{\lambda_{\mathrm{ls}}}} x \Leftrightarrow \forall y \prec x \; \exists z \prec y \; \limsup z \leq a$. The implication "$\Leftarrow$" is Theorem 4.3(d). In order to prove "$\Rightarrow$" suppose that $a \in \lim_{\mathcal{O}_{\lambda_{\mathrm{ls}}}} x$, $y \prec x$ and $\limsup z \not\leq a$, for each subsequence $z \prec y$. By ($\hbar$), there is a $\limsup$-stable sequence $z \prec y$. Then the set $K = \{n \in \omega : z_n \leq a\}$ is finite, since otherwise we would have $\limsup \langle z_n : n \in K \rangle \leq a$. Thus w.l.o.g we can suppose that $z_n \not\leq a$ for each $n \in \omega$. By Lemma 5.4 we have

$$\overline{\{z_n : n \in \omega\}} = (\limsup z)\uparrow \cup \bigcup_{n\in\omega} z_n\uparrow .$$

Thus $a \in O = \mathbb{B} \setminus \overline{\{z_n : n \in \omega\}} \in \mathcal{O}_{\lambda_{\mathrm{ls}}}$ and, since $O \cap \{z_n : n \in \omega\} = \emptyset$, we have $a \notin \lim_{\mathcal{O}_{\lambda_{\mathrm{ls}}}} z$. A contradiction, because $z \prec x$ and $a \in \lim_{\mathcal{O}_{\lambda_{\mathrm{ls}}}} x$. $\square$

Example 6.2 If $\mathbb{B}$ is a ccc complete Boolean algebra such that forcing by $\mathbb{B}$ produces new reals, then, by Fact 3.4 and Theorems 3.5, 5.1 and 6.1, the convergences $\lambda_{\mathrm{s}}, \lambda_{\mathrm{ls}}$ and λ_{li} are weakly topological, but not topological. In particular this holds for the Cohen algebra Borel$(2^\omega)/\mathcal{M}$ and random algebra Borel$(2^\omega)/\mathcal{Z}$, where $\mathcal{M}$ and $\mathcal{Z}$ are the σ-ideals of meager and measure-zero Borel sets, respectively.

In the sequel, using the following lemma, we show that, on complete Boolean algebras belonging to a large class, the convergence λ_{ls} is not weakly-topological.

Lemma 6.3 Let $\mathbb{B}$ be a complete Boolean algebra, $x = \langle x_n : n \in \omega \rangle$ a sequence in $\mathbb{B}$ and $\tau_x = \{\langle \check{n}, x_n \rangle : n \in \omega\}$ the corresponding $\mathbb{B}$-name for a real. Then

(a) If A is an infinite subset of ω and $f_A : \omega \to A$ is the corresponding increasing bijection, then $\| |\tau_x \cap \check{A}| = \check{\omega} \| = \limsup x \circ f_A$.

(b) The following conditions are equivalent:

(i) $\forall f \in \omega^{\uparrow\omega} \; \exists g \in \omega^{\uparrow\omega} \; \limsup x \circ f \circ g = 0$;

(ii) $\forall y \prec x \; \exists z \prec y \; \limsup z = 0$;

(iii) $\forall A \in [\omega]^\omega \; \exists B \in [A]^\omega \; \| |\tau_x \cap \check{B}| = \check{\omega} \| = 0$.

Proof. (a) Since $A = \{f_A(n) : n \in \omega\}$ and f_A is a bijection, $\limsup x \circ f_A = \bigwedge_{k\in\omega} \bigvee_{n\geq k} x_{f_A(n)} = \|\forall k \in \check\omega\ \exists n \geq k\ f_A(n) \in \tau_x\| = \||\tau_x \cap \check A| = \check\omega\|$.

(b) The equivalence of (i) and (ii) is obvious.

(i) $\Rightarrow$ (iii) Let $A \in [\omega]^\omega$. By (i), there is $g \in \omega^{\uparrow\omega}$ such that $\limsup x \circ f_A \circ g = 0$. Clearly $B = f_A[g[\omega]] \in [A]^\omega$ and $f_B = f_A \circ g$ so, by (a), $\||\tau_x \cap \check B| = \check\omega\| = \limsup x \circ f_A \circ g = 0$.

(iii) $\Rightarrow$ (i) Let $f \in \omega^{\uparrow\omega}$ and $A = f[\omega]$. By (iii), there is $B \in [A]^\omega$ such that $\||\tau_x \cap \check B| = \check\omega\| = 0$. Since $f^{-1}[B] \in [\omega]^\omega$, there exists an increasing bijection $g : \omega \to f^{-1}[B]$. From $B \subset f[\omega]$ it follows that $f[g[\omega]] = f[f^{-1}[\omega]] = B$. So, by (a), $\limsup x \circ f \circ g = \||\tau_x \cap f[g[\omega]]\check{}\,| = \check\omega\| = 0$ and (i) is proved. $\Box$

We remind the reader that a set $\mathcal{T} \subset [\omega]^\omega$ is called a **base matrix tree** iff $\langle \mathcal{T}, {}^*\supset\rangle$ is a tree of height $\mathfrak{h}$ and $\mathcal{T}$ is a dense set in the pre-order $\langle [\omega]^\omega, \subset^*\rangle$. By a theorem of Balcar, Pelant and Simon (see [4]), such a tree always exists. Clearly the levels of a base matrix tree $\mathcal{T}$ are maximal almost disjoint families and maximal chains in $\mathcal{T}$ are towers.

Theorem 6.4 If $\mathbb{B}$ is a complete Boolean algebra satisfying $1 \Vdash_{\mathbb{B}} (\mathfrak{h}^V)\check{} < \mathfrak{t}$ and $\mathrm{cc}(\mathbb{B}) > 2^{\mathfrak{h}}$, then λ_{ls} is not a weakly-topological convergence on $\mathbb{B}$.

Proof. Let $\mathcal{T}$ be a base matrix tree and $\mathrm{Br}(\mathcal{T})$ the set of all maximal branches of $\mathcal{T}$. Since the levels of $\mathcal{T}$ are of size $\leq \mathfrak{c}$ and the height of $\mathcal{T}$ is $\mathfrak{h}$, for $\kappa = |\,\mathrm{Br}(\mathcal{T})|$ we have $\kappa \leq \mathfrak{c}^{\mathfrak{h}} = 2^{\mathfrak{h}}$ and we take an enumeration $\mathrm{Br}(\mathcal{T}) = \{T_\alpha : \alpha < \kappa\}$. Since $1 \Vdash (\mathfrak{h}^V)\check{} < \mathfrak{t}$, for each $\alpha < \kappa$ we have $1 \Vdash |\check T_\alpha| < \mathfrak{t}$ and, consequently, $1 \Vdash \exists X \in [\check\omega]^{\check\omega}\ \forall B \in \check T_\alpha\ X \subset^* B$ so, by the Maximum Principle (see [9, p. 226]) there is a name $\sigma_\alpha \in V^{\mathbb{B}}$ such that

$$1 \Vdash \sigma_\alpha \in [\check\omega]^{\check\omega} \wedge \forall B \in T_\alpha\ \ \sigma_\alpha \subset^* B. \tag{11}$$

Let $\{b_\alpha : \alpha < \kappa\}$ be a maximal antichain in $\mathbb{B}$. By the Mixing lemma (see [9, p. 226]) there is a name $\tau \in V^{\mathbb{B}}$ such that

$$\forall \alpha < \kappa\ \ b_\alpha \Vdash \tau = \sigma_\alpha, \tag{12}$$

and, clearly, $1 \Vdash \tau \in [\check\omega]^{\check\omega}$. Let us define $x_n = \|\check n \in \tau\|$, $n \in \omega$. Then for the corresponding name $\tau_x = \{\langle \check n, x_n\rangle : n \in \omega\}$ we have

$$1 \Vdash \tau = \tau_x. \tag{13}$$

Claim 1. $0 \notin \lambda^*_{\mathrm{ls}}(x)$.

Proof of Claim 1: We prove that $\neg \forall y \prec x\ \exists z \prec y\ \limsup z = 0$ that is, by Lemma 6.3(b), $\exists A \in [\omega]^\omega\ \forall B \in [A]^\omega\ \||\tau_x \cap \check B| = \check\omega\| > 0$. In fact, we show more:

$$\forall B \in [\omega]^\omega\ \ \||\tau_x \cap \check B| = \check\omega\| > 0. \tag{14}$$

Let $B \in [\omega]^\omega$. Since $\mathcal{T}$ is a dense subset of $\langle [\omega]^\omega, \subset^* \rangle$ there is $C \in \mathcal{T}$ such that $C \subset^* B$. Let T_α be a branch in $\mathcal{T}$ such that $C \in T_\alpha$. Then, by (12) and (13) we have $b_\alpha \Vdash \tau_x = \sigma_\alpha$, and by (11) $1 \Vdash \sigma_\alpha \subset^* C$, so $b_\alpha \leq \| |\tau_x \cap \check{B}| = \check{\omega} \|$.

Claim 2. $0 \in \lim_{\mathcal{O}_{\lambda_{\mathrm{ls}}}}(x)$.

Proof of Claim 2: On the contrary, suppose that there are $F \in \mathcal{F}_{\lambda_{\mathrm{ls}}}$ and $A \in [\omega]^\omega$ such that $0 \notin F$ and $\{x_n : n \in A\} \subset F$. Since $\mathcal{T}$ is dense in $\langle [\omega]^\omega, \subset^* \rangle$, there is $C \in \mathcal{T}$ such that $C \subset^* A$ and, clearly, there is $\alpha < \kappa$ such that $C \in T_\alpha$. T_α is a tower of type $\lambda \leq \mathfrak{h}$, so $T_\alpha = \{B_\xi : \xi < \lambda\}$, where $B_\zeta \subsetneq^* B_\xi$, for $\xi < \zeta < \lambda$. Let $C = B_{\xi_0}$ and, for $n \in \omega$, let

$$D_n = B_{\xi_0+n} \setminus B_{\xi_0+n+1}.$$

By Lemma 6.3(a), for each $n \in \omega$ we have $\| |\tau_x \cap \check{D}_n| = \check{\omega} \| = \limsup x \circ f_{D_n}$. Since $D_n \subset^* A$, almost all members of the sequence $x \circ f_{D_n}$ are elements of F and, by Theorem 4.4(I), $\| |\tau_x \cap \check{D}_n| = \check{\omega} \| \in F$. So, by the same theorem, $\limsup \| |\tau_x \cap \check{D}_n| = \check{\omega} \| \in F$. Since $\limsup \| |\tau_x \cap \check{D}_n| = \check{\omega} \| = \| |\tau_x \cap \check{D}_n| = \check{\omega}$ for infinitely many $n \in \omega \|$, we will obtain a contradiction when we prove that

$$\| |\tau_x \cap \check{D}_n| = \check{\omega} \text{ for infinitely many } n \in \omega \| = 0. \tag{15}$$

Let G be a $\mathbb{B}$-generic filter over V. Then there exists $\beta < \kappa$ such that $b_\beta \in G$ and, by (11),(12) and (13),

$$(\tau_x)_G \subset^* B, \text{ for each } B \in T_\beta. \tag{16}$$

First, if $\beta = \alpha$ then, by (16), $|(\tau_x)_G \cap D_n| < \omega$, for each $n \in \omega$.
Second, if $\beta \neq \alpha$, we have two cases.

Case 1: $\exists E \in T_\beta \ \forall n \in \omega \ E \subset^* B_{\xi_0+n}$. Then $(\tau_x)_G \subset^* E$ and for each $n \in \omega$ we have $|(\tau_x)_G \cap D_n| < \omega$.

Case 2: $\forall E \in T_\beta \ \exists n \in \omega \ E \not\subset^* B_{\xi_0+n}$. Then, since $\mathcal{T}$ is a tree, there is the $\subset^*$-maximum of the set $T_\beta \setminus T_\alpha$, say E' and, by the assumption, there is $n_0 \in \omega$ such that $B_{\xi_0+n_0} \subset^* E'$ or $|B_{\xi_0+n_0} \cap E'| < \omega$. Since $E' \notin T_\alpha$, $B_{\xi_0+n_0} \subset^* E'$ is impossible, so $|B_{\xi_0+n_0} \cap E'| < \omega$ and, hence, $|B_{\xi_0+n} \cap E'| < \omega$, for each $n \geq n_0$. Since $(\tau_x)_G \subset^* E'$ and $D_n \subset B_{\xi_0+n}$, we have $|(\tau_x)_G \cap D_n| < \omega$, for all $n \geq n_0$.

Thus $|(\tau_x)_G \cap D_n| < \omega$, for all but finitely many $n \in \omega$ and (15) is true. □

The following example shows that there are very simple Boolean algebras such that the question "Is the convergence λ_{ls} on $\mathbb{B}$ weakly topological?" does not have an answer in ZFC.

Example 6.5 The statement "The convergence λ_{ls} on the collapsing algebra $\mathbb{B} = \mathrm{ro}({}^{<\omega}\omega_2)$ is weakly topological" is independent of ZFC. Since $\omega_2^{<\omega} = \omega_2$, the algebra $\mathbb{B}$ is ω_3-cc and collapses ω_2 to ω in each generic extension.

If in the ground model V we have $2^\omega = \omega_1$ and $2^{\omega_1} = \omega_2$ (in particular, if $V \models$ GCH) then in V we have $\mathfrak{h} = \omega_1$, $\mathrm{cc}(\mathbb{B}) = \omega_3 > \omega_2 = 2^{\mathfrak{h}}$ and $1 \Vdash_{\mathbb{B}} |(\mathfrak{h}^V)\check{}\,| = \check{\omega}$. Thus, by Theorem 6.4, the convergence λ_{ls} on $\mathbb{B}$ is not weakly topological.

On the other hand, if in V we have $\mathfrak{t} \geq \omega_3$ (in particular, if $V \models \mathrm{MA} + \mathfrak{c} \geq \omega_3$), then $\mathbb{B}$ is $\mathfrak{t}$-cc and, hence, satisfies condition $(\hbar)$ which, by Theorem 6.1, implies that the convergence λ_{ls} on $\mathbb{B}$ is weakly topological.

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